January 20, 2011

CONTINUOUS RATIONAL FUNCTIONS ON REAL AND p-ADIC VARIETIES

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A *rational function* on $\mathbb{R}^n$ is a quotient of two polynomials

$$f(x_1,\dots,x_n) := \frac{p(x_1,\dots,x_n)}{q(x_1,\dots,x_n)}.$$

Strictly speaking, a rational function f is not really a function on $\mathbb{R}^n$ in general since it is defined only on the dense open set where $q \neq 0$. Nonetheless, even if q vanishes at some points of $\mathbb{R}^n$, it can happen that there is an everywhere defined continuous function f^* that agrees with f at all points where f is defined. Such an f^* is unique. For this reason, we identify f with f^* and call f itself a *continuous rational function* on $\mathbb{R}^n$. For instance,

$$\frac{p(x_1,\dots,x_n)}{x_1^{2m}+\dots+x_n^{2m}}$$

is a continuous rational function on $\mathbb{R}^n$ if every monomial in p has degree $> 2m$.

The above definition of continuous rational functions makes sense on any real algebraic variety X, as long as the open set where f is defined is dense in X in the Euclidean topology. This condition always holds on smooth varieties, and, more generally, if the singular set is contained in the Euclidean closure of the smooth locus.

The aim of this note is to consider three basic problems on continuous rational functions.

Question 1. Let X be a real algebraic variety and Z a closed subvariety.

(1) Let f be a continuous rational function defined on X. Is the restriction $f|_Z$ a rational function on Z?
(2) Let g be a continuous rational function defined on Z. Can one extend it to a continuous rational function on X?
(3) Which systems of linear equations

$$\textstyle\sum_j f_{ij}(x_1,\dots,x_n)\cdot y_j = g_i(x_1,\dots,x_n) \qquad i=1,\dots,m$$

have continuous rational solutions where the g_i, f_{ij} are polynomials (or rational functions) on X.

Section 1 contains some examples that I found interesting. Section 2 contains the proofs of a few results. The answer to Question 1.1 is always yes if X is smooth (8) but not in general. This leads to the introduction of the notion of *hereditarily rational functions.* These have the good properties that one would expect based on the smooth case. In Theorem 9 this concept is used to give a complete answer to Question 1.2. I do not have a full answer to Question 1.3.

All these results extend to p-adic and other topological fields, see (13).

1. Examples

The following example shows that the answer to Questions 1.1–2 is not always positive.

Example 2. Consider the surface S and the rational function f given by

$$S := \big(x^3 - (1+z^2)y^3 = 0\big) \subset \mathbb{R}^3 \quad \text{and} \quad f(x,y,z) := \tfrac{x}{y}.$$

We claim that

1. (1) S is a real analytic submanifold of $\mathbb{R}^3$,
2. (2) f is defined away from the z-axis,
3. (3) f extends to a real analytic function on S, yet
4. (4) the restriction of f to the z-axis is not rational and
5. (5) f can not be extended to a continuous rational function on $\mathbb{R}^3$.

Proof. Note that

$$x^3 - (1+z^2)y^3 = \big(x - (1+z^2)^{1/3}y\big)\big(x^2 + (1+z^2)^{1/3}xy + (1+z^2)^{2/3}y^2\big).$$

The first factor defines S as a real analytic submanifold of $\mathbb{R}^3$. The second factor only vanishes along the z-axis which is contained in S. Therefore $x - y\sqrt[3]{1+z^2}$ vanishes on S, hence

$$f|_S = \tfrac{x}{y}|_S = \sqrt[3]{1+z^2}|_S \quad \text{and so} \quad f(0,0,z) = \sqrt[3]{1+z^2}.$$

Assume finally that F is a continuous rational function on $\mathbb{R}^3$ whose restriction to S is f. Then F and f have the same restrictions to the z-axis. We show below that the restriction of any continuous rational function F defined on $\mathbb{R}^3$ to the z-axis is a rational function. Thus $F|_{z-\text{axis}}$ does not equal $\sqrt[3]{1+z^2}$, a contradiction.

To see the claim, write $F = p(x,y,z)/q(x,y,z)$ where p,q are polynomials. We may assume that they are relatively prime. Since F is continuous everywhere, x can not divide q. Hence $F|_{(x=0)} = p(0,y,z)/q(0,y,z)$. By canceling common factors, we can write this as $F|_{(x=0)} = p_1(y,z)/q_1(y,z)$ where p_1, q_1 are relatively prime polynomials. As before, y can not divide q_1, hence $F|_{z-\text{axis}} = p_1(0,z)/q_1(0,z)$ is a rational function. (Note that we seemingly have not used the continuity of F: for any rational function $f(x,y,z)$ the above procedure defines a rational function on the $z-$axis. However, if we use x,y in reverse order, we could get a different rational function. This happens, for instance, for $f = x^2/(x^2+y^2)$.)

In the above example, the problems arise since S is not normal. However, the key properties (2.4–5) can also be realized on a normal hypersurface.

Example 3. Consider

$$X := \big((x^3 - (1+t^2)y^3)^2 + z^6 + y^7 = 0\big) \subset \mathbb{R}^4 \quad \text{and} \quad f(x,y,z,t) := \tfrac{x}{y}.$$

We easily see that the singular locus is the t-axis and so X is normal. Let us blow up the t-axis. There is one relevant chart, where $x_1 = x/y, y_1 = y, z_1 = z/y$. We get the smooth 3-fold

$$X' := \big((x_1^3 - (1+t^2))^2 + z_1^6 + y_1 = 0\big) \subset \mathbb{R}^4.$$

Each point $(0,0,0,t)$ has only 1 preimage in X', given by $\big(\sqrt[3]{1+t^2}, 0, 0, t\big)$ and the projection $\pi : X' \to X$ is a homeomorphism. Thus X is a topological manifold, but it is not a differentiable submanifold of $\mathbb{R}^4$.

Since $f \circ \pi = x_1$ is a regular function, we conclude that f extends to a continuous function on X and $f(0,0,0,t) = \sqrt[3]{1+t^2}$. Thus, as before, f can not be extended to a continuous rational function on $\mathbb{R}^4$.

For any $m \geq 1$ we get the similar examples

$$X_m := \big((x^3-(1+t^2)y^3)^2 + z_1^6 + \cdots + z_m^6 + y^7 = 0\big) \subset \mathbb{R}^{3+m} \quad \text{and} \quad f := \tfrac{x}{y}.$$

For $m \gg 1$ the X_m have rational, even terminal singularities.

Next we turn to Question 1.3 for a single equation

$$\textstyle\sum_i f_i(\mathbf{x}) \cdot y_i = g(\mathbf{x}), \tag{$*$}$$

where g and the f_i are polynomials in the variables $\mathbf{x} = (x_1, \ldots, x_n)$. Such equations have a solution where the y_i are rational functions provided not all of the f_i are identically zero. The existence of a solution where the y_i are continuous functions is studied in [FK11] and [Kol10].

One could then hope to prove that if there is a continuous solution then there is also a continuous rational solution. In [FK11, Sec.2] we proved that if there is a continuous solution then there is also a continuous semialgebraic solution. The next example shows, however, that in general there is no continuous rational solution.

Example 4. We claim that the linear equation

$$x_1^3 x_2 \cdot y_1 + \big(x_1^3 - (1+x_3^2)x_2^3\big) \cdot y_2 = x_1^4 \tag{4.1}$$

has a continuous semialgebraic solution but no continuous rational solution.

A continuous semialgebraic solution is given by

$$y_1 = (1+x_3^2)^{1/3} \quad \text{and} \quad y_2 = \frac{x_1^3}{x_1^2 + (1+x_3^2)^{1/3}x_1x_2 + (1+x_3^2)^{2/3}x_2^2}. \tag{4.2}$$

(Note that $x_1^2 + (1+x_3^2)^{1/3}x_1x_2 + (1+x_3^2)^{2/3}x_2^2 \geq \frac{1}{2}(x_1^2+x_2^2)$, so $|y_2| \leq 2x_1$ and it is indeed continuous.) A solution by rational functions is $y_1 = x_1/x_2$ and $y_2 = 0$.

To see that (4.1) has no continuous rational solution, restrict any solution to the semialgebraic surface $S := (x_1 - (1+x_3^2)^{1/3}x_2 = 0)$. Since $x_1^3 - (1+x_3^2)x_2^3$ is identically zero on S, we conclude that $y_1|_S = x_1^4/(x_1^3x_2)|_S = (x_1/x_2)|_S$. The latter is equal to $\sqrt[3]{1+x_3^2}$, thus

$$y_1|_{x_3-\text{axis}} = \sqrt[3]{1+x_3^2}$$

which is not a rational function. As we saw in Example 2, this implies that y_1 is not a rational function.

Next we give an example of a continuous rational function f on $\mathbb{R}^3$ and of an irreducible algebraic surface $S \subset \mathbb{R}^3$ such that $f|_S$ is zero on a Zariski dense open set yet $f|_S$ is not identically zero.

Example 5. Consider the surface $S := \big(x^2+y^2z^2-y^3 = 0\big) \subset \mathbb{R}^3$. Its singular locus is the z-axis and the Euclidean closure $S^* \subset S$ of the smooth locus is the preimage of the inside of the parabola $(y \geq z^2) \subset \mathbb{R}^2_{yz}$ under the coordinate projection. Consider the rational function

$$f(x,y,z) := z^2 \cdot \frac{x^2+y^2z^2-y^3}{x^2+y^2z^2+y^4}.$$

We see that f vanishes on S^* and its only possible discontinuities are along the z-axis. To analyze its behavior there, rewrite it as

$$f = z^2 - y(1+y) \cdot \frac{y^2z^2}{x^2 + y^2z^2 + y^4}.$$

The fraction is bounded by 1, hence f is everywhere continuous and $f(0,0,z) = z^2$.

If $g(z)$ is any rational function without real poles then $g(z)f(x,y,z)$ vanishes on S^* and its restriction to the z-axis is $z^2g(z)$.

(The best known example of a surface with a Zariski dense open set which is not Euclidean dense is the Whitney umbrella $W := (x^2 = y^2z) \subset \mathbb{R}^3$. We can take $W_1 := (x = y = 0)$ and $W_2 := W$. The Euclidean closure of $W_2 \setminus W_1$ does not contain the "handle" $(x = y = 0, z < 0)$.

In this case, a continuous rational function is determined by its restriction to $W_2 \setminus W_1$. The Euclidean closure of $W_2 \setminus W_1$ contains the half line $(x = y = 0, z \geq 0)$, and a rational function on a line is determined by its restriction to any interval.)

The next example shows the two natural ways of pulling back continuous rational functions by a birational morphism can be different.

Example 6 (Two pull-backs)**.** Let $\pi : X' \to X$ be a proper, birational morphism of real varieties. If f is a continuous rational function on X, then one can think of the composite $f \circ \pi$ in two different ways.

First, $f \circ \pi$ is the composite of two continuous maps, hence it is a continuous function. Second, one can view $f \circ \pi$ as a rational function on X', which may have a continuous extension to X'.

Tho following example shows that these two notions can be different.

Take $X = \mathbb{R}^2$ with $f = x^3/(x^2+y^2)$. Note that $f(0,0) = 0$. Blow up (x^3, x^2+y^2) to obtain $X' \subset \mathbb{R}^2 \times \mathbb{RP}^1$. The first projection $\pi : X' \to X$ is an isomorphism away from the origin and $\pi^{-1}(0,0) \cong \mathbb{RP}^1$.

The first interpretation above gives a continuous function $f \circ \pi$ which vanishes along $\pi^{-1}(0,0)$.

The second interpretation views $f \circ \pi$ as a rational map $X' \dashrightarrow \mathbb{RP}^1$, which is in fact regular. Its restriction to $\pi^{-1}(0,0)$ is an isomorphism $\pi^{-1}(0,0) \cong \mathbb{RP}^1$.

This confusion is possible only because $X'_0 := \pi^{-1}\bigl(\mathbb{R}^2 \setminus (0,0)\bigr)$ is not Euclidean dense in X'. Its Euclidean closure contains only one point of $\pi^{-1}(0,0) \cong \mathbb{RP}^1$. The two versions of $f \circ \pi$ agree on X'_0, hence also on its Euclidean closure, but not everywhere.

2. Hereditarily rational functions

In order to get some results, we have to restrict to those continuous rational functions for which Question 1.1 has a positive answer. First we show that this is always the case on smooth varieties and then we prove that for such functions Question 1.2 also has a positive answer.

Definition 7. Let X be a real algebraic variety and f a continuous function on X. We say that f is *hereditarily rational* if every irreducible, real subvariety $Z \subset X$ has a Zariski dense open subvariety $Z^0 \subset Z$ such that $f|_{Z^0}$ is a regular function on Z^0. (A function f is *regular at* $x \in X$ if one can write $f = p/q$ where p, q are polynomials and $q(x) \neq 0$. It is called *regular* if it is regular at every point of X.)

Examples 2 and 3 show that not every continuous rational function is hereditarily rational.

If f is rational, there is a Zariski dense open set $X^0 \subset X$ such that $f|_{X^0}$ is regular. If f is continuous and hereditarily rational, we can repeat this process with the restriction of f to $X \setminus X^0$, and so on. Thus we conclude that a continuous function f is hereditarily rational iff there is a sequence of closed subvarieties $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_m = X$ such that for $i = 0, \dots, m$ the restriction of f to $X_i \setminus X_{i-1}$ is regular.

If it is convenient, we can also assume that each $X_i \setminus X_{i-1}$ is smooth of pure dimension i.

The pull-back of a hereditarily rational function by any morphism is again a (continuous and) hereditarily rational function.

The next result shows that on a smooth variety every continuous rational function is hereditarily rational.

Proposition 8. *Let X be a real algebraic variety and Z an irreducible subvariety that is not contained in the singular locus of X. Let f be a continuous rational function on X. Then there is a Zariski dense open subset $Z^0 \subset Z$ such that $f|_{Z^0}$ is a regular function.*

Proof. By replacing X with a suitable open subvariety, we may assume that X and Z are both smooth.

Assume first that Z has codimension 1. Then the local ring $\mathcal{O}_{X,Z}$ is a principal ideal domain [Sha94, Sec.II.3.1]; let $t \in \mathcal{O}_{X,Z}$ be a defining equation of Z. We can write $f = t^m u$ where $m \in \mathbb{Z}$ and $u \in \mathcal{O}_{X,Z}$ is a unit. Here $m \geq 0$ since f does not have a pole along Z, hence f is regular along a Zariski dense open subset $Z^0 \subset Z$. Thus $f|_{Z^0}$ is a regular function.

If Z has higher codimension, note that Z is a local complete intersection at its smooth points [Sha94, Sec.II.3.2]. That is, there is a sequence of subvarieties $Z \supset Z_0 \subset Z_1 \subset \cdots \subset Z_m = X_0 \subset X$ where each Z_i is a smooth hypersurface in Z_{i+1} for $i = 0, \dots, m-1$ and Z_0 (resp. X_0) is open and dense in Z (resp. X). We can thus restrict f to Z_{m-1}, then to Z_{m-2} and so on, until we get that $f|_{Z_0}$ is regular. (As we noted in (4), for any rational function f he above procedure defines a regular function $f|_{Z_0}$, but it depends on the choice of the chain $Z_1 \subset \cdots \subset Z_m$.) $\square$

The following result shows that hereditarily rational functions constitute the right class for Question 1.2.

Theorem 9. *Let Z be a real algebraic variety and f a continuous rational function on Z. The following are equivalent.*

1. *f is hereditarily rational.*
2. *For every real algebraic variety X that contains Z as a closed subvariety, f extends to a (continuous and) hereditarily rational function F on X.*
3. *For every real algebraic variety X that contains Z as a closed subvariety, f extends to a continuous rational function F on X.*
4. *Let X_0 be a smooth real algebraic variety that contains Z as a closed subvariety. Then f extends to a continuous rational function F_0 on X_0.*

Proof. It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) holds by (8). Thus we need to show that (1) $\Rightarrow$ (2).

Note first that a regular function on a closed subvariety Z of a real algebraic variety X always extends to a regular function on the whole variety. (This follows, for instance, from [BCR98, 4.4.5] or see (14) for a more general argument.)

We reduce the hereditarily rational case to the regular case by a sequence of blow-ups.

We can embed X into a smooth real algebraic variety X'. If we can extend f to X' then its restriction to X gives the required extension. Thus we may assume to start with that X is smooth (or even that $X = \mathbb{R}^N$ for some N).

Let $\emptyset = Z_{-1} \subset Z_0 \subset \cdots \subset Z_m = Z$ be a sequence of closed subvarieties as in (7). By induction on j we construct composites of blow ups with smooth centers $\Pi_j : X_j \to X$ such that

(5) Π_j^{-1} is an isomorphism over $X \setminus Z_{j-1}$,

(6) $E_j := \Pi_j^{-1}(Z_{j-1})$ is a simple normal crossing divisor,

(7) $E_j + W_j \subset X_j$ is a simple normal crossing subvariety (11), where W_j denotes the birational transform of Z_j,

(8) $f \circ \Pi_j$ is a regular function on $E_j + W_j$.

At the end we have $\Pi_{m+1} : X_{m+1} \to X$ with exceptional set $E_{m+1} = \Pi_{m+1}^{-1}(Z_m) = \Pi_{m+1}^{-1}(Z)$ such that $f \circ \Pi_{m+1}$ is a regular function on E_{m+1}. Thus $f \circ \Pi_{m+1}$ extends to a regular function F_{m+1} on X_{m+1}. By construction F_{m+1} is constant on the fibers of $E_{m+1} \to Z$ and the other fibers of Π_{m+1} consist of one point each since $(X_{m+1} \setminus E_{m+1}) \cong (X \setminus Z)$. Thus by (10), F_{m+1} descends to a continuous rational function F on X. Since X is smooth, F is hereditarily rational by (8).

Now to the inductive argument. We will repeatedly use Hironaka-type resolution theorems; see [Kol07, Chap.III] for references and relatively short proofs.

We can start induction with $X_0 := X$. Assume that we already have $\Pi_j : X_j \to X$. Then $f \circ \Pi_j|_{W_j}$ is a continuous function which is regular on $(W_j \setminus E_j) \cong (Z_j \setminus Z_{j-1})$. Since W_j is smooth, $W_j \setminus E_j$ is Euclidean dense in W_j, hence (unlike in Example 6) we can identify $f \circ \Pi_j|_{W_j}$ with the corresponding rational function.

Thus there is a sequence of blow-ups $q_j : W_j' \to W_j$ whose centers are smooth and lie over $W_j \cap E_j$ such that W_j' is smooth and $f_j' := \bigl(f \circ \Pi_j|_{W_j}\bigr) \circ q_j$ is a regular map to $\mathbb{P}^1$. On the other hand, f_j' is also the pull-back of the continuous function $f|_{W_j}$, hence it is a regular function. We can perform the same blow-ups on X_j to get X_j' such that $W_j' \subset X_j'$.

At each step we blow up over the locus E_j where f is known to be regular. Thus the pull-back of f stays regular over the preimage of E_j. After a further sequence of blow-ups we may assure the conditions (9.5–7).

Finally, (9.8) follows from (11). □

Lemma 10. *Let $p : Y \to X$ be a proper birational morphism of real algebraic varieties. Assume that on both of them, the singular set is contained in the Euclidean closure of the smooth locus. Then composing with p gives a one-to-one correspondence between continuous rational functions on X and continuous rational functions on Y that are constant on the fibers of p.* □

11. Let X be a smooth variety. A closed subvariety $Z = \cup_j Z_j \subset X$ has *simple normal crossing* at a point $p \in X$ if one can choose local coordinates $x_1, \dots, x_m$ at p such that, in a neighborhood of p, each irreducible component Z_j can be defined by equations $\{x_i = 0 : i \in I(j)\}$ for some $I(j) \subset \{1, \dots, m\}$. If this holds at every point, we say that Z is a simple normal crossing subvariety.

Let $Z = \cup_j Z_j \subset X$ be a simple normal crossing subvariety and f a function on Z. We claim that the following are equivalent.

(1) f is regular on Z.
(2) f is continuous on Z and $f|_{Z_j}$ is regular for every j.

It is clear that (1) $\Rightarrow$ (2). To see the converse, it is sufficient to prove that f is real analytic at each point. Choose local coordinates as above. We can uniquely write $f|_{Z_j}$ as a power series in the variables $\{x_i : i \in \{1, \dots, m\} \setminus I(j)\}$. Over $Z_j \cap Z_i$ we get the same power series, thus we can write $f = \sum_J a_J x^J$ where $a_J = 0$ if x^J does not appear in any of the $f|_{Z_j}$ and a_J equals the coefficient of x^J in $f|_{Z_j}$ whenever it is not zero.

12. It would be desirable to find an elementary proof of (9), one that does not rely so much on resolution of singularities.

One key question is the following. For any real algebraic variety W one would like to construct a proper birational morphism $W' \to W$ such that the smooth locus of W' is Euclidean dense in W'. Resolution of singularities provides such a W' but I do not know any simple construction.

13 (Varieties over other topological fields)**.** Let K be any topological field. The K-points $X(K)$ of any K-variety inherit from K a topology, called the K-topology. One can then consider rational functions f on X that are continuous on $X(K)$. This does not seem to be a very interesting notion in general, unless K satisfies the following *density property.*

(DP) If X is smooth, irreducible and $\emptyset \neq U \subset X$ is Zariski open then $U(K)$ is dense in $X(K)$ in the K-topology.

Note that if (DP) holds for all smooth curves then it holds for all varieties. It is easy to see that if K is not discrete and the implicit function theorem holds over K then K has the above density property. Such examples are the p-adic fields $\mathbb{Q}_p$, their finite extensions and, more generally, quotient fields of complete local rings.

The results of this section all hold over such fields of characteristic 0. (The last assumption is needed only for resolution of singularities).

The only step that needs additional proof is the assertion that every regular function on a subvariety extends to the whole variety. This is proved by a slight modification of the usual arguments that apply when k is algebraically closed [Sha94, Sec.I.3.2] or real closed [BCR98, 3.2.3].

Lemma 14 (Extending regular functions)**.** *Let k be a field, X an affine k-variety and $Z \subset X$ a closed subvariety. Let f be a rational function on Z that is regular at all points of $Z(k)$. Then there is a rational function F on X that is regular at all points of $X(k)$ and such that $F|_Z = f$.*

Proof. If k is algebraically closed, then f is regular on Z hence it extends to a regular function on X.

If k is not algebraically closed, then, as a auxiliary step, we claim that there are homogeneous polynomials $G(x_1, \dots, x_r)$ in any number of variables whose only zero on k^r is $(0, \dots, 0)$.

Indeed, if k is real, then we can take $G = \sum x_i^2$. By a theorem of Artin–Schreier, if k is not real closed, then it has finite extensions L/k whose degree is arbitrary large (cf. [Jac80, Sec.11.7]). If $c_1, \dots, c_r$ is a k-basis of L, then $G = \operatorname{Norm}_{L/k}(\sum x_i c_i)$ works.

Now we construct the extension of f as follows.

For every $z \in Z(k)$ we can write $f = p_z/q_z$ where $q_z(z) \neq 0$. After multiplying both p_z, q_z with a suitable polynomial, we can assume that p_z, q_z are regular on Z and then extend them to regular functions on X. By assumption, $\bigcap_{z\in Z(k)}(q_z = 0)$ is disjoint from $Z(k)$. Choose finitely many $z_1, \dots, z_m \in Z(k)$ such that

$$\textstyle\bigcap_{i=1}^{m}\bigl(q_{z_i} = 0\bigr) = \bigcap_{z\in Z(k)}\bigl(q_z = 0\bigr).$$

Let $q_{m+1}, \dots, q_r$ be defining equations of $Z \subset X$. Set $q_i := q_{z_i}$ and $p_i := p_{z_i}$ for $i = 1, \dots, m$ and $p_i = q_i$ for $i = m+1, \dots, r$. Write (non-uniquely) $G = \sum G_i x_i$ and finally set

$$F := \frac{\sum_{i=1}^{r} G_i(q_1, \dots, q_r)p_i}{G(q_1, \dots, q_r)}.$$

Since the q_i have no common zero on $X(k)$, we see that F is regular at all points of $X(k)$.

Along Z, $p_i = fq_i$ for $i = 1, \dots, m$ by construction and for $i = m+1, \dots, r$ since then both sides are 0. Thus

$$F|_Z := \frac{\sum_{i=1}^{r} G_i(q_1, \dots, q_r)fq_i}{G(q_1, \dots, q_r)}|_Z = f \cdot \frac{\sum_{i=1}^{r} G_i(q_1, \dots, q_r)q_i}{G(q_1, \dots, q_r)}|_Z = f. \quad \square$$

Acknowledgments. I thank Ch. Fefferman, M. Jarden, F. Mangolte and B. Totaro for useful comments and questions. Partial financial support was provided by the NSF under grant number DMS-0758275.

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